ISRAEL JOURNAL OF MATHEMATICS. Vol. 56. No. 1. 1986

FITTING CLASSES $\mathscr{F}$ SUCH THAT ALL FINITE GROUPS HAVE $\mathscr{F}$-INJECTORS

BY

M. J. IRANZO AND F. PÉREZ MONASOR

Departamento de Algebra y Fundamentos, Facultad de Ciencias Matemáticas, Universidad de Valencia, Spain

ABSTRACT

Let $\mathscr{F}$ be an homomorph and Fitting class such that $E_z\mathscr{F} = \mathscr{F}$. In this paper we prove that if all $\mathscr{F}$-constrained groups have $\mathscr{F}$-injectors, then all groups have $\mathscr{F}$-injectors. In particular if $\mathscr{F}$ is a class of quasinilpotent groups containing the nilpotent groups, then every group has $\mathscr{F}$-injectors.

Introduction. Notation

All groups considered throughout this paper will be finite. We denote by $\mathcal{N}$ the class of nilpotent groups and by $\mathcal{N}^*$ the class of quasinilpotent groups, i.e.

$$\mathcal{N}^* = \{G \mid G = F(G)L(G) = F^*(G)\}.$$

(For the basic properties of quasinilpotent groups and of the $\mathcal{N}^*$-radical $F^*(G)$ of a group G, the reader is referred to ([4]) X.ϕ 13; we shall use these properties without further reference.) The concept of semisimple group is taken from Gorenstein–Walter's paper ([3]). We denote by $L(G)$ the semisimple radical of G, $L(G)$ is sometimes called the layer of G. A group G is $\mathcal{N}$-constrained if $C_G(F(G)) \leqq F(G)$ ([7]) and a group G is $\mathcal{N}$-constrained if and only if $L(G) = 1$ ([8]).

Throughout the paper $\mathscr{F}$ means an homomorph of Fitting such that $E_z\mathscr{F} = \mathscr{F}$ (i.e. if $G/Z(G) \in \mathscr{F}$ then $G \in \mathscr{F}$). A group G is $\mathscr{F}$-constrained if $C_G(G_{\mathscr{F}}) \leqq G_{\mathscr{F}}$, where $G_{\mathscr{F}}$ is the $\mathscr{F}$-radical of G. The class of $\mathscr{F}$-constrained groups is a Fitting class denoted by $X_{\mathscr{F}}$ and a group G is $\mathscr{F}$-constrained if and only if $L(G) \in \mathscr{F}$ ([5]).

In 1971 A. Mann proved that an $\mathcal{N}$-constrained group has a unique conjugacy class of $\mathcal{N}$-injectors. Blessenohl and Laue proved that all groups have a unique conjugacy class of $\mathcal{N}^*$-injectors. It is well known that all groups are $\mathcal{N}^*$-

Received October 8, 1985

constrained. In ([5]) we proved that if $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{N}^*$, then all $\mathcal{F}$-constrained groups have $\mathcal{F}$-injectors.

The aim of this paper is mainly to prove the following:

THEOREM. (a) *Let G be a group, X an $\mathcal{F}$-injector of $L(G)$. If $N_G(X)$ has $\mathcal{F}$-injectors, these are also $\mathcal{F}$-injectors of G.*

(b) *If all $\mathcal{F}$-constrained groups have $\mathcal{F}$-injectors then all groups have $\mathcal{F}$-injectors.*

Before proving the theorem we give the following results.

LEMMA 1. *Let G_i be a non-abelian simple group, $i = 1, \dots, n$ and $G = G_1 \times \cdots \times G_n$, then G contains $\mathcal{F}$-injectors which are the product of the $\mathcal{F}$-injectors of the factors.*

PROOF. By induction over n. First we suppose $n = 2$. Let V_1, V_2 be $\mathcal{F}$-injectors of G_1, G_2 respectively. Let V be an $\mathcal{F}$-maximal subgroup of G containing $V_1 \times V_2$. Since the only subnormal subgroups of G are $1, G_1, G_2, G$, it is clear that V is an $\mathcal{F}$-injector of G. Moreover:

$$D = V_1 \times V_2 = (V \cap G_1) \times (V \cap G_2) \leqq V.$$

On the other hand, $VG_1/G_1 \in \mathcal{F}$ and this group contains $(V \cap G_2)G_1/G_1 \cong V \cap G_2$ that is an $\mathcal{F}$-injector of G_2. Hence $VG_1 = (V \cap G_2)G_1$ and so: $|V| = |V \cap G_1|\,|V \cap G_2|$ and $V = V_1 \times V_2$.

Now suppose $n > 2$. Let V_i be an $\mathcal{F}$-injector of G_i, $i = 1, \dots, n$ and let V be an $\mathcal{F}$-maximal of G containing $V_1 \times V_2 \times \cdots \times V_n$.

Put $N = G_1 \times G_2 \times \cdots \times G_{n-1}$, then $V \cap N = V_1 \times V_2 \times \cdots \times V_{n-1}$ by inductive hypothesis. On the other hand $VN/N \in \mathcal{F}$ and this group contains $(V \cap G_n)N/N \cong V \cap G_n$ that is an $\mathcal{F}$-injector of $G_n \cong G/N$, hence: $VN = (V \cap G_n)N$ and so $|V| = |V \cap G_n|\,|V \cap N|$. Therefore $V = V_1 \times \cdots \times V_n$. Finally since every subnormal subgroup of G is a product of some G_i, now we have, by inductive hypothesis, that $V = V_1 \times V_2 \times \cdots \times V_n$ is an $\mathcal{F}$-injector of G.

LEMMA 2. *A group G possesses $\mathcal{F}$-injectors if and only if $G/Z(G)$ possesses $\mathcal{F}$-injectors.*

PROOF. Let H be an $\mathcal{F}$-injector of G and $G^*/Z(G) \trianglelefteq\trianglelefteq G/Z(G)$, then: $H/Z(G) \cap G^*/Z(G) = (H \cap G^*)/Z(G)$. Since $\mathcal{F}$ is extensible by central subgroups, it follows that $(H \cap G^*)/Z(G)$ is $\mathcal{F}$-maximal subgroup of $G^*/Z(G)$. Therefore $H/Z(G)$ is an $\mathcal{F}$-injector of $G/Z(G)$.

Conversely, assume that $H/Z(G)$ is an $\mathcal{F}$-injector of $G/Z(G)$ and $G^* \trianglelefteq\trianglelefteq G$. Let $H \cap G^* \leqq F \leqq G^*$, $F \in \mathcal{F}$, then we have:

$$(H \cap G^*)Z(G)/Z(G) \leqq FZ(G)/Z(G) \leqq G^*Z(G)/Z(G) \trianglelefteq\trianglelefteq G/Z(G);$$

since $FZ(G)/Z(G) \in \mathcal{F}$, it follows that: $(H \cap G^*)Z(G) = FZ(G)$ and thus: $F = F \cap (H \cap G^*)Z(G) = (H \cap G^*)(F \cap Z(G)) = H \cap G^*$.

COROLLARY 1. *If G is a semisimple group, then G contains $\mathcal{F}$-injectors.*

PROOF. Since $G/Z(G)$ is a direct product of non-abelian simple groups, this is a consequence of Lemmas 1 and 2.

COROLLARY 2. *Let G be a semisimple group $G = G_1G_2$ where G_i is a semisimple group, $i = 1,2$ and $[G_1, G_2] = 1$. If J is an $\mathcal{F}$-injector of G, then: $J = (J \cap G_1)(J \cap G_2)$.*

PROOF. By Lemma 2, $J/Z(G)$ is an $\mathcal{F}$-injector of $G/Z(G)$ and this is a direct product of non-abelian simple groups. Since $G/Z(G) = (G_1Z(G)/Z(G)) \times (G_2Z(G)/Z(G))$, we can apply Lemma 1 to obtain:

$$J/Z(G) = (J/Z(G) \cap G_1Z(G)/Z(G))(J/Z(G) \cap G_2Z(G)/Z(G))$$

and hence:

$$J = (J \cap G_1Z(G))(J \cap G_2Z(G)) = (J \cap G_1)(J \cap G_2)Z(G) = (J \cap G_1)(J \cap G_2)$$

since $Z(G) = Z(G_1)Z(G_2)$.

PROOF OF THE THEOREM. (a) Let J be an $\mathcal{F}$-injector of $N_G(X)$. First we prove that J is an $\mathcal{F}$-maximal subgroup of G.

Let H be an $\mathcal{F}$-subgroup of G such that $J \leqq H \leqq G$. Then $H \cap L(G) = X$ and hence $H \leqq N_G(X)$ and $J = H$.

Using induction on $|G|$, we can now prove that J is an $\mathcal{F}$-injector of G.

Let G^* be a maximal normal subgroup of G. We can consider two cases:

Case 1. $L(G) \leqq G^*$

In this case: $L(G) = L(G^*)$, X is an $\mathcal{F}$-injector of $L(G^*)$ and $J \cap G^*$ is an $\mathcal{F}$-injector of $N_G(X) \cap G^* = N_{G^*}(X)$. By inductive hypothesis it follows that $J \cap G^*$ is an $\mathcal{F}$-injector of G^*.

Case 2. $L(G) \nleqq G^*$

In this case $L(G) = L(G^*)R$ where R is a semisimple normal subgroup of G and $[L(G) \cap G^*, R] = 1$.

Notice that:

$$[G^*, R] \leqq [G^*, L(G)] \leqq G^* \cap L(G).$$

The three-subgroups lemma together with the perfectness of R yield that $[G^*, R] = 1$.

Now, by Corollary 2 we obtain:

$$X = (X \cap L(G^*))(X \cap R)$$

and so

$$N_{G^*}(X \cap L(G^*)) \leqq N_{G^*}(X) \leqq N_{G^*}(X \cap L(G^*)).$$

Since $J \cap G^*$ is an $\mathcal{F}$-injector of $N_{G^*}(X) = N_{G^*}(X \cap L(G^*))$, using inductive hypothesis we obtain that $J \cap G^*$ is an $\mathcal{F}$-injector of G^*.

(b) Let G be a counterexample of minimal order. We can suppose that $L(G) \notin \mathcal{F}$, because if $L(G) \in \mathcal{F}$ then G would be an $\mathcal{F}$-constrained group.

Let X be an $\mathcal{F}$-injector of $L(G)$. If $N_G(X) < G$ then $N_G(X)$ contains $\mathcal{F}$-injectors and by part (a) it follows that G contains $\mathcal{F}$-injectors, a contradiction. Therefore $N_G(X) = G$ and so $X \lhd L(G)$ hence $L(G) = XR$, where R is a non-trivial semisimple normal subroup of $L(G)$ and $[X, R] = 1$. Then $X \cap R = Z(R)$ is an $\mathcal{F}$-injector of R. Whence, using Lemma 2, it follows that 1 is an $\mathcal{F}$-injector of $R/Z(R)$ and so $R/Z(R) = 1$, thus R is trivial, a contradiction.

Corollary 3. *If $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{N}^*$, then all groups have $\mathcal{F}$-injectors. In particular, all groups have $\mathcal{N}$-injectors.*

Proof. By ([5]) we know that all $\mathcal{F}$-constrained groups have $\mathcal{F}$-injectors and so it suffices to apply part (b) of the above theorem.

The particular case $\mathcal{F} = \mathcal{N}$ has been recently obtained by P. Förster ([2]).

Remarks.

(1) If $\mathcal{F}$ verifies the hypothesis of part (a) of the theorem, and G is a group, then every $\mathcal{F}$-injector of $L(G)$ is contained in an $\mathcal{F}$-injector of G.

(2) Let G be a group such that every $\mathcal{F}$-injector of $L(G)$ is contained in an $\mathcal{F}$-injector of G. Suppose that G possesses a unique conjugacy class of $\mathcal{F}$-injectors, then G is an $\mathcal{F}$-constrained group.

Indeed, let I_1, I_2 be $\mathcal{F}$-injectors of $L(G)$ and V_1, V_2 $\mathcal{F}$-injectors of G containing I_1, I_2 respectively, then there exists $g \in G$ such that $V_2 = V_1^g$ and so: $I_1^g = V_1^g \cap L(G) = V_2 \cap L(G) = I_2$. Let p, q be prime divisors of $|G|$ with $p \neq q$ and P, Q p, q-Sylow subgroups of $L(G)$ respectively. Then there exist I_1, I_2 $\mathcal{F}$-injectors of $L(G)$ such that $P \leqq I_1$ and $Q \leqq I_2$. Thus $P^g \leqq I_1^g = I_2$ for a $g \in G$. With this method we can obtain that $L(G) \in \mathcal{F}$ and then G is an $\mathcal{F}$-constrained group.

(3) Let V_1, V_2 be $\mathcal{N}$-injectors of a group G, such that $V_1 \cap C_G(F(G))$ and $V_2 \cap C_G(F(G))$ are conjugated in G. Then V_1 and V_2 are conjugated in G.

In fact, there exists $g \in G$ such that $V_1^g \cap C_G(F(G)) = V_2 \cap C_G(F(G)) = I$, then V_1^g and V_2 are $\mathcal{N}$-maximal subgroups of G containing $IF(G)$. Clearly $C_G(IF(G)) = C_{C_G(F(G))}(I) \leqq I$ and using Lausch's theorem ([6]) we obtain that V_1 and V_2 are conjugated in G.

(4) Let π be a set of prime numbers. The classes

$$\chi_\pi = \{G \mid G = F(G)O_\pi(L(G))\}$$

are examples of homomorphs of Fitting $\mathcal{F}$ verifying $E_z\mathcal{F} = \mathcal{F}$. These classes of quasinilpotent groups contain the nilpotent groups.

(5) Let $\mathcal{H}$ be a Fitting class. Assume: (i) $\mathcal{N} \subseteq \mathcal{H} \subseteq \mathcal{N}^*$ and (ii) whenever $G \in \mathcal{H}$ it follows that $G/Z \in \mathcal{H}$ for every $Z \leqq Z(G)$. Then $\mathcal{H}$ is an homomorph.

References

1. D. Blessenohl and H. Laue, *Fittingklassen endlicher Gruppen, in denen gewisse Haupfaktoren einfach sind*, J. Algebra **56** (1979), 516–532.
2. P. Förster, *Nilpotent injectors in finite groups*, Bull. Austral. Math. Soc., to appear.
3. D. Gorenstein and J. Walter, *The π-layer of a finite group*, Ill. J. Math. **15** (1971), 555–564.
4. B. Huppert and N. Blackburn, *Finite Groups III*, Springer-Verlag, Berlin, 1982.
5. M. J. Iranzo and F. Pérez Monasor, *$\mathcal{F}$-constraint with respect to a Fitting class*, Arch. Math., to appear.
6. H. Lausch, *Conjugacy classes of maximal nilpotent subgroups*, Isr. J. Math. **47** (1984), 29–31.
7. A. Mann, *Injectors and normal subgroups of finite groups*, Isr. J. Math. **9** (1971), 554–558:
8. F. Pérez Monasor, *Grupos finitos separados respecto de una Formación de Fitting*, Rev. Acad. Ciencias de Zaragoza, serie 2ª, XXVIII, **3** (1973), 253–301.